1968

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[redacted]

Submitted in partial fulfillment of the
requirements for the degree of

MASTER OF SCIENCE
WITH MAJOR IN MATHEMATICS

from the

NAVAL POSTGRADUATE SCHOOL
March 1968

ABSTRACT

The possibility of generalizing König's Theorem to functions of more than one variable is investigated. The generalized Taylor series expansion of a function of several variables is introduced, and the ratio of coefficients in the expansion is defined using Fréchet derivatives. It is shown that for a particular example of a function of two variables the generalized König's Theorem holds. The theorem is then shown to hold for a large class of functions.

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ACKNOWLEDGMENT

I wish to thank Professor Hugo M. Martinez of the Naval Postgraduate School for his advice and assistance in the preparation of the Thesis.

I. Introduction

In this thesis we develop some of the concepts which may be useful in generalizing König's theorem from functions of one variable to functions of several real variables. König's theorem relates the coefficients in the Taylor series expansion of a function to the singularity of the function.

The development of this possible generalization of König's theorem is divided into five sections. The first section, the introduction, gives the background on König's theorem and a definition of the ratio of two successive coefficients in the generalized Taylor series expansion is developed. The second section introduces Fréchet differentiation and points out that the Fréchet derivatives are actually the coefficients in the Taylor series. Using the Fréchet derivatives and their associated properties, the third section proves that it is possible to define the ratio of two successive coefficients in the expansion as was presented in the first section. The fourth section illustrates by means of an example that König's theorem holds when we use the defined ratio. The conclusions are presented in the fifth section.

In this section we will give the background on König's theorem needed for the possible generalization we will develop in the rest of the paper. This is done by first presenting König's theorem and an outline of its proof, then showing an extension of the theorem which is important in numerical analysis. The Taylor series expansion for functions of several real variables is presented and a definition for the ratio of two successive coefficients is developed.

König's theorem is proven by Householder who proceeds as follows: [7] Theorem: Consider any function of the complex variable z, f(z), whose Taylor series expansion

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_\nu z^\nu + \cdots$$

converges in some circle about the origin. Suppose that within this circle f(z) has one and only one zero, α, which is simple. Let g(z) be another function which is analytic throughout the circle and $g(\alpha) \neq 0$. Let

$$g/f \equiv h(z) = h_0 + h_1 z + h_2 z^2 + \cdots + h_\nu z^\nu + \cdots ,$$

then

$$\lim_{\nu \to \infty} h_\nu / h_{\nu+1} = \alpha .$$

Proof of König's theorem: The expansion

$$h(z) = h_0 + h_1 z + h_2 z^2 + \cdots + h_\nu z^\nu + \cdots$$

converges for all $|z| < |\alpha|$, while the expansion

$$(\alpha - z) h(z) \equiv F(z) = k_0 + k_1 z + k_2 z^2 + \cdots + k_\nu z^\nu + \cdots$$

converges throughout the circle. Then for $|z| < |\alpha|$

$$(\alpha - z)(h_0 + h_1 z + h_2 z^2 + \cdots + h_\nu z^\nu + \cdots)$$
$$= k_0 + k_1 z + k_2 z^2 + \cdots + k_\nu z^\nu + \cdots .$$

On equating coefficients of powers of z we get

$$\alpha h_0 = k_0,$$

$$-h_0 + \alpha h_1 = k_1,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$-h_{\nu-1} + \alpha h_\nu = k_\nu .$$

Multiplying these equations by 1, α, α^2, ... and adding gives

$$\alpha^{\nu+1} h_\nu = k_0 + k_1 \alpha + \cdots + k_\nu \alpha^\nu .$$

Now let

$$F_\nu(z) \equiv k_0 + k_1 z + k_2 z^2 + \cdots + k_\nu z^\nu \equiv F(z) - R_{\nu+1}(z),$$

then

$$h_\nu = \alpha^{-\nu-1} F_\nu(\alpha) = \alpha^{-\nu-1}[F(\alpha) - R_{\nu+1}(\alpha)],$$

and

$$h_\nu / h_{\nu+1} = \alpha F_\nu(\alpha) / F_{\nu+1}(\alpha) .$$

However F is analytic at α, and the series expansion for F(z) converges for $z = \alpha$. Therefore

$$\lim_{\nu \to \infty} h_\nu / h_{\nu+1} = \alpha . \qquad \text{Q. E. D.}$$

It should be noted that König's theorem could have been stated differently. Instead of starting with the function f(z) with a

zero at α, the theorem could have begun with the function h which has a simple pole at α. The reason for the use of two functions, f and g, is because the main application of the theorem is for finding the roots of the function f.

König's theorem has an important extension for a function f(z) which has n simple zeros, $\alpha_1, \alpha_2, \ldots, \alpha_n$, within some circle about the origin. The extended theorem states that these zeros are actually the roots of the equation

$$c_0 z^n + c_1 z^{n-1} + \cdots + c_{n-1} z + c_n = 0.$$

The coefficients $c_0, c_1, \ldots, c_n$ represent the cofactors of the powers of z in the determinant

$$\begin{vmatrix} h_\nu & h_{\nu-1} & \cdot & \cdot & \cdot & h_{\nu-n+1} & z^n \\ h_{\nu+1} & h_\nu & \cdot & \cdot & \cdot & h_{\nu-n+2} & z^{n-1} \\ \vdots & \vdots & & & & \vdots & \vdots \\ h_{\nu+n} & h_{\nu+n-1} & \cdot & \cdot & \cdot & h_{\nu+1} & 1 \end{vmatrix} = 0$$

for large values of ν. Again the h_ν are defined as coefficients in the expansion of g/f, where g is any analytic function which is not zero at any of the zeros of f.

The extension allows one to find the roots $\alpha_1, \alpha_2, \alpha_3, \ldots$ of a function, if these roots are contained in circles of increasing size about the origin. In other words, there exists a circle which contains α_1 alone, a larger circle which contains just α_1 and α_2, a still larger circle which contains α_1, α_2, and α_3,

and so forth. Having found α_1, we can apply the extension of König's theorem and find the product $\alpha_1\alpha_2$. Set

$$H_\nu^{(2)} = \begin{vmatrix} h_\nu & h_{\nu-1} \\ h_{\nu+1} & h_\nu \end{vmatrix}$$

then

$$\lim_{\nu\to\infty} H_\nu^{(2)}/H_{\nu+1}^{(2)} = \alpha_1\alpha_2$$

this give α_2 since α_1 is known. Again if

$$H_\nu^{(3)} = \begin{vmatrix} h_\nu & h_{\nu-1} & h_{\nu-2} \\ h_{\nu+1} & h_\nu & h_{\nu-1} \\ h_{\nu+2} & h_{\nu+1} & h_\nu \end{vmatrix},$$

then

$$\lim_{\nu\to\infty} H^{(3)}_\nu / H^{(3)}_{\nu+1} = \alpha_1\alpha_2\alpha_3 .$$

Continuing in this manner the other roots can be found.

To generalize König's theorem to functions of several real variables we must first look at a more generalized Taylor series expansion. [2,3] In the case of a function of two real variables, h(x,y)

$$h(x,y) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} \left.\frac{\partial^n h}{\partial x^{n-k}\partial y^k}\right|_{0,0} x^{n-k}y^k .$$

For the extension to a function h of m variables, x^1, x^2, ..., x^m, it is convenient to introduce the operator

$$D^n_{i_1 i_2 \cdots i_n} h(0) = \left. \frac{\partial^n h}{\partial x^{i_1} \partial x^{i_2} \cdots \partial x^{i_n}} \right|_{X=0} .$$

Using summation convention on the i's (i_1, i_2, ..., i_n = 1,2,...,m), Taylor's series expansion is then

$$h(x^1, x^2, \cdots, x^m) = \sum_{n=0}^{\infty} \frac{1}{n!} D^n_{i_1 i_2 \cdots i_n} h(0) x^{i_1} x^{i_2} \cdots x^{i_n} .$$

Letting V be m-dimensional, we define

$$h^{(n)}(0) : \underbrace{V \times V \times \cdots \times V}_{n\text{-times}} \to \text{Reals}$$

as

$$h^{(n)}(0)(X_1, X_2, \cdots, X_n) = D^n_{i_1 i_2 \cdots i_n} h(0) x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$$

where

$$X_j = (x_j^1, x_j^2, \cdots, x_j^m) .$$

If $X_1 = X_2 = \ldots = X_n = X$, then the n-tuple $(X_1, X_2, \ldots, X_n)$ will be denoted as X^n. We can then write the Taylor series expansion for h(X) expanded about X = 0 as

$$h(X) = \sum_{n=0}^{\infty} \frac{1}{n!} h^{(n)}(0) X^n .$$

Using the notation developed above, we can now state the generalization of König's theorem as follows: Consider any

function h(X), of m real variables, whose expansion about the origin

$$h(X) = \sum_{n=0}^{\infty} \frac{1}{n!} h^{(n)}(0) X^{n}$$

converges in some spherical neighborhood about the origin. Suppose that h(X) has only one singularity α or that α is the singularity nearest the origin, then

$$\lim_{\nu \to \infty} \frac{1}{n!} h^{(n)}(0) \Big/ \frac{1}{(n+1)!} h^{(n+1)}(0) = \alpha .$$

It should be emphasized that the formal coefficients in this Taylor series representation of h(X) are multilinear functionals and hence that the ratio between two successive ones as needed for the generalization of König's theorem must be suitably defined. In particular we want such a ratio

$$\frac{1}{n!} h^{(n)}(0) \Big/ \frac{1}{(n+1)!} h^{(n+1)}(0)$$

to be an element of our space V so that we can generate a sequence of elements in V that converge to the singularity of h nearest the origin.

It is natural to define this ratio in the following sense:

$$\frac{\frac{1}{n!} h^{(n)}(0)}{\frac{1}{(n+1)!} h^{(n+1)}(0)} = y$$

in V if for all X in V, it is true that

$$\frac{1}{n!} h^{(n)}(0)(\underbrace{X, X, \cdots, X}_{n\text{-times}}) = \frac{1}{(n+1)!} h^{(n+1)}(0)(\underbrace{X, X, \cdots, X}_{n\text{-times}}, Y).$$

One would anticipate that such a Y may not exist or even that it may not be unique. Accordingly, the strict equality of above is weakened as an equality in the uniform sense. Thus, Y will be defined as the vector which minimizes the integral

$$\int_{\|X\|=1} \left[\frac{1}{n!} h^{(n)}(0) X^n - \frac{1}{(n+1)!} h^{(n+1)}(0) X^n Y \right]^2 dX.$$

It will be shown in Section 3 that if $h^{(n+1)}(0)$ is not the zero multi-functional, then there exist a unique Y which minimizes this integral. Clearly, if the minimum is zero, then strict equality will hold.

The derivatives we have used are known as Fréchet derivatives which will next be discussed for purposes of establishing a wider background necessary for extension of König's theorem other than to the functions of several variables as is the restricted aim in this paper.

II. Differentiation

In functional analysis a number of definitions of the differential of a function have been used. [6,13] Perhaps the simplest is the "weak" or Gateaux differential which is simply the "variation" used in the calculus of variations. [12]

In order to define the Gateaux differential, we consider an operator f whose domain X and range Y are real normed linear spaces. [14] If we let x_0 be an interior point of X and let h, an element of X, be arbitrary, then if the limit

$$\delta f(x_0, h) = \lim_{t \to 0} \frac{1}{t} \left[f(x_0 + th) - f(x_0) \right]$$

exists, it is called the Gateaux differential of f at x_0 with increment h. f is said to be Gateaux-differentiable at x_0 if f has a Gateaux differential at x_0 for every h in X; and f is called Gateaux-differentiable on a subset of X if f is Gateaux-differentiable at every x_0 in the subset.

This can be made clear by an example. Consider X as the ordinary Euclidean plane (ξ_1, ξ_2) and Y as the real line. Then $T = F(\xi_1, \xi_2)$ is Gateaux-differentiable at a point (ξ_1', ξ_2') if

$$\delta F(x, h) = \lim_{t \to 0} \frac{1}{t} \left[F(\xi_1' + th_1, \xi_2' + th_2) - F(\xi_1', \xi_2') \right]$$

exists for every $h = (h_1, h_2)$ in X. This is equivalent to requiring that the direction derivative of $F(\xi_1, \xi_2)$ exists at (ξ_1', ξ_2') for every direction vector h in X.

Some of the properties of the Gateaux differential are:

(1) If f_1 and f_2 are Gateaux-differentiable so is $f = \alpha f_1 + \beta f_2$ for any real α, β and

$$\delta f(x_0,h) = \alpha \delta f_1(x_0,h) + \beta \delta f(x_0,h).$$

(2) If $\delta f(x_0,h)$ exists, so does $\delta f(x_0, \alpha h)$ for any real α and

$$\delta f(x_0, \alpha h) = \alpha \delta f(x_0,h)$$

so that $\delta f(x_0,h)$ is homogeneous of degree one in h.

It should be noted that the Gateaux differential is not necessarily linear in h, but if this is the case, $\delta f(x_0,h) = A_{x_0}(h)$, where A_{x_0} is a linear operator from X into Y. The linear operator A_{x_0} is sometimes called the Gateaux derivative of f at x_0. Even if the Gateaux differential is linear in h, it need not be continuous in h.

However, the Gateaux differential does not have many of the usual properties associated with the total differentiation for functions of two or more real variables, and we shall be interested in the generalization of the total differential. There are several interpretations of the total differential, but the standard definition of a differential of a function is due to Fréchet. [1,15]

Assuming the same properties of f stated previously, f is said to be Fréchet-differentiable at an interior point x_0 in X

if it has a Gateaux differential which is linear and continuous in h and if

$$\lim_{h \to 0} \frac{1}{\|h\|} \left\| f(x_0+h) - f(x_0) - \delta f(x_0,h) \right\| = 0.$$

$\delta f(x_0,h)$ is then called the Fréchet differential of f at x_0 with increment h. f is Fréchet-differentiable on a subset of X if it is Fréchet-differentiable at every point in the subset. [5,10,12]

In addition to the properties mentioned previously for the Gateaux differential, the Fréchet differential, $\delta f(x_0,h)$, is additive in h, i.e.

$$\delta f(x_0, h_1 + h_2) = \delta f(x_0, h_1) + \delta f(x_0, h_2)$$

and hence linear in h because of the additional homogeneity property.

In investigating the relationship between h and the Fréchet differential $\delta f(x_0,h)$, we find that by definition the correspondence is a bounded linear operator A_{x_0} depending on x_0. What we are now interested is is the correspondence between x_0 and A_{x_0} which is actually an operator from X into the normed linear space $[X \to Y]$ of bounded linear operators from X to Y. This operator is called the Fréchet derivative, f' or $f^{(1)}$, of f. Thus, $f'(x_0) = A_{x_0}$.

A linear operator, A, is Fréchet-differentiable only if it is defined and bounded on X. This is because a linear transformation is continuous if and only if it is bounded. If A'

exists, $A'(x_o) = A$, so that A' is defined on X also. For consider

$$A'(x_0)x = \lim_{t \to 0} \frac{A(x_0 + tx) - A(x_0)}{t}$$

$$= \lim_{t \to 0} \frac{A(x_0) + tA(x) - A(x_0)}{t}$$

$$= \lim_{t \to 0} A(x) = A(x).$$

If an operator f on X to Y is Fréchet-differentiable, then f' may be Fréchet-differentiable, also. If this is the case the Fréchet derivative of f' is called the second Fréchet derivative of f and denoted by f'' or $f^{(2)}$, and it is an operator from X into the normed linear space of bounded linear operators from X into $[X \to Y]$.

In other words, if the n^{th} Fréchet derivative exists, we have

$$f^{(1)}: X \to [X \to Y]$$

$$f^{(2)}: X \to [X \to [X \to Y]$$

$$\vdots$$

$$f^{(n)}: X \to [\underbrace{X \to [X \to \cdots \to [X}_{n\text{-times}} \to Y]\cdots].$$

For example, consider a function of two variables, $h(x^1,x^2)$ where $X = (x^1,x^2)$ and $Y = (y^1,y^2)$ are in E_2, then some of the Fréchet derivatives of h are

$$h^{(2)}(0)(X,X) = h^{(2)}(0)X^2 = \left.\frac{\partial^2 h}{\partial x^{1^2}}\right|_{0,0} x^1x^1$$

$$+ 2\left.\frac{\partial^2 h}{\partial x^1 \partial x^2}\right|_{0,0} x^1x^2 + \left.\frac{\partial^2 h}{\partial x^{2^2}}\right|_{0,0} x^2x^2 ,$$

$$h^{(3)}(0)(X,X,X) = h^{(3)}(0)X^3 = \left.\frac{\partial^3 h}{\partial x^{1^3}}\right|_{0,0} x^1x^1x^1$$

$$+ 3\left.\frac{\partial^3 h}{\partial x^{1^2}\partial x^2}\right|_{0,0} x^1x^1x^2 + 3\left.\frac{\partial^3 h}{\partial x^1 \partial x^{2^2}}\right|_{0,0} x^1x^2x^2$$

$$+ \left.\frac{\partial^3 h}{\partial x^{2^3}}\right|_{0,0} x^2x^2x^2$$

$$= \left.\frac{\partial^3 h}{\partial x^1\partial x^1\partial x^1}\right|_{0,0} x^1x^1x^1 + \frac{\partial^3 h}{\partial x^1\partial x^1\partial x^2} x^1x^1x^2$$

$$+ \left.\frac{\partial^3 h}{\partial x^1\partial x^2\partial x^1}\right|_{0,0} x^1x^2x^1 + \left.\frac{\partial^3 h}{\partial x^2\partial x^1\partial x^1}\right|_{0,0} x^2x^1x^1$$

$$+ \left.\frac{\partial^3 h}{\partial x^2\partial x^2\partial x^1}\right|_{0,0} x^2x^2x^1 + \left.\frac{\partial^3 h}{\partial x^2\partial x^1\partial x^2}\right|_{0,0} x^2x^1x^2$$

$$+ \left.\frac{\partial^3 h}{\partial x^1\partial x^2\partial x^2}\right|_{0,0} x^1x^2x^2 + \left.\frac{\partial^3 h}{\partial x^2\partial x^2\partial x^2}\right|_{0,0} x^2x^2x^2 ,$$

$$h^{(3)}(0)(X,X,Y) = h^{(3)}(0)X^2Y = \left.\frac{\partial^3 h}{\partial x^1 \partial x^1 \partial x^1}\right|_{0,0} x^1 x^1 y^1$$

$$+ \left.\frac{\partial^3 h}{\partial x^1 \partial x^1 \partial x^2}\right|_{0,0} x^1 x^1 x^2 + \left.\frac{\partial^3 h}{\partial x^1 \partial x^2 \partial x^1}\right|_{0,0} x^1 x^2 y^1$$

$$+ \left.\frac{\partial^3 h}{\partial x^2 \partial x^1 \partial x^1}\right|_{0,0} x^2 x^1 y^1 + \left.\frac{\partial^3 h}{\partial x^2 \partial x^2 \partial x^1}\right|_{0,0} x^2 x^2 y^1$$

$$+ \left.\frac{\partial^3 h}{\partial x^2 \partial x^1 \partial x^2}\right|_{0,0} x^2 x^1 y^2 + \left.\frac{\partial^3 h}{\partial x^1 \partial x^2 \partial x^2}\right|_{0,0} x^1 x^2 y^2$$

$$+ \left.\frac{\partial^3 h}{\partial x^2 \partial x^2 \partial x^2}\right|_{0,0} x^2 x^2 y^2 .$$

A property of the Fréchet derivative that is useful in the next section is that it is a multilinear functional, if Y is the scalar field of X. [8] A functional is multilinear if it is linear in each of the arguments separately. To show this property is true, consider

$$W \in [\underbrace{X \to [X \to \cdots \to [X}_{n\text{-times}} \to Y] \cdots] .$$

Given any x_1 in X, obviously

$$W(x_1) \in [\underbrace{X \to [X \to \cdots \to [X}_{(n-1)\text{-times}} \to Y] \cdots],$$

and W is linear in the argument x_1, since W evaluated at x_1 is a linear map. Then take any x_2 in X, obviously

$$W(x_1, x_2) \in [\underbrace{X \to [X \to \cdots \to [X}_{(n-2)-\text{times}} \to Y] \cdots]$$

and $W(x_1)$ is linear in the argument x_2, since $W(x_1)$ evaluated at x_2 is a linear map. Continuing in this manner we find that

$$W(x_1, x_2, \cdots, x_n) \in Y$$

and W is linear in each argument. Hence the Fréchet derivative is a multilinear functional.

It should be noted that in some references Gateaux and Fréchet differentiability is referred to simply as G - differentiable or F - differentiable.

At this point it should be remembered that the reason for the development of this generalized differentiation is to find the ratio of two successive coefficients in the generalized Taylor series expansion. The expansion can now be expressed in terms of Fréchet derivatives as

$$h(X) = \sum_{n=0}^{\infty} \frac{1}{n!} h^{(n)}(0) X^n$$

where $h^{(n)}(0)$ is the n^{th} Fréchet derivative of the function h evaluated at the origin and $(\overbrace{X, X, \ldots, X}^{n\text{-times}})$ is denoted as X^n.

Now that we have the terms in the Taylor series expansion expressed as Fréchet derivatives, we will use the properties of

these derivatives in the next section in defining the ratio of two successive coefficients.

III. Evaluation of the Ratio of Coefficients

Consider the Taylor series expansion for a function of several real variables,

$$h(X) = \sum_{n=0}^{\infty} \frac{1}{n!} h^{(n)}(0) X^n ,$$

where $h^{(n)}(0)$ is the n^{th} Fréchet derivative of the function h evaluated at the origin and $(\overbrace{X,X,\ldots,X}^{n\text{-times}})$ is denoted as X^n. In Section I the ratio of two successive coefficients was defined as

$$\frac{\frac{1}{n!} h^{(n)}(0)}{\frac{1}{(n+1)!} h^{(n+1)}(0)} = Y$$

where Y is the vector that minimizes the integral

$$\int_{\|X\|=1} \left[\frac{1}{n!} h^{(n)}(0) X^n - \frac{1}{(n+1)!} h^{(n+1)}(0) X^n Y \right]^2 dX .$$

The purpose of this section is to show that there exists a unique Y which minimizes this integral. This is possible since as was noted in the last section Fréchet derivatives are multi-linear functionals and we have a special case of the following lemma.

Lemma: Given a k^{th} order multi-linear functional, f, and a non-zero $(k+1)^{st}$ order multi-linear functional, g, defined on the

same finite dimensional space, then there exists a unique Y in the vector space such that

$$\int_{\|X\|=1} [f(X_1,X_2,\cdots,X_K) - g(X_1,X_2,\cdots,X_K,Y)]^2 \, dX$$

is minimized. The notation $\|X\| = 1$ means that the norm of each X vector is equal to 1.

Proof: Using the summation convention, let

$$f(X_1,X_2,\cdots,X_K) = a_{i_1,i_2,\cdots,i_K} x_1^{i_1} x_2^{i_2} \cdots x_K^{i_K}$$

and

$$g(X_1,X_2,\cdots,X_K,Y) = b_{i_1,i_2,\cdots,i_K,i_{K+1}} x_1^{i_1} x_2^{i_2} \cdots x_K^{i_K} y^{i_{K+1}}$$

where $X_j^{i_j}$ represents the i_j th element of the vector X_j.

Then,

$$\int_{\|X\|=1} [f(X_1,X_2,\ldots,X_K) - g(X_1,X_2,\cdots,X_K,Y)]^2 \, dX$$

$$= \int_{\|X\|=1} [a_{i_1,i_2,\ldots,i_K} x_1^{i_1} x_2^{i_2} \cdots x_K^{i_K} - b_{i_1,i_2,\ldots,i_K,i_{K+1}} x_1^{i_1} x_2^{i_2} \cdots x_K^{i_K} y^{i_{K+1}}]^2 \, dX$$

$$\geq 0$$

$$= \int_{\|X\|=1} (a_{i_1,i_2,\ldots,i_K} x_1^{i_1} x_2^{i_2} \cdots x_K^{i_K} - b_{i_1,i_2,\ldots,i_K,i_{K+1}} x_1^{i_1} x_2^{i_2} \cdots x_K^{i_K} y^{i_{K+1}}) \cdot$$

$$(a_{j_1,j_2,\cdots,j_K} x_1^{j_1} x_2^{j_2} \cdots x_K^{j_K} - b_{j_1,j_2,\ldots,j_K,j_{K+1}} x_1^{j_1} x_2^{j_2} \cdots x_K^{j_K} y^{j_{K+1}}) \, dX$$

$$\geq 0$$

$$= \int_{\|X\|=1} a_{i_1,i_2,\dots,i_k} x_1^{i_1} x_2^{i_2} \cdots x_k^{i_k} a_{j_1,j_2,\dots,j_k} x_1^{j_1} x_2^{j_2} \cdots x_k^{j_k} \, dX$$

$$+ \int_{\|X\|=1} -a_{i_1,i_2,\dots,i_k} x_1^{i_1} x_2^{i_2} \cdots x_k^{i_k} b_{j_1,j_2,\dots,j_k,j_{k+1}} x_1^{j_1} x_2^{j_2} \cdots x_k^{j_k} y^{j_{k+1}} \, dX$$

$$+ \int_{\|X\|=1} -a_{j_1,j_2,\dots,j_k} x_1^{j_1} x_2^{j_2} \cdots x_k^{j_k} b_{i_1,i_2,\dots,i_k,i_{k+1}} x_1^{i_1} x_2^{i_2} \cdots x_k^{i_k} y^{i_{k+1}} \, dX$$

$$+ \int_{\|X\|=1} b_{i_1,i_2,\dots,i_k,i_{k+1}} x_1^{i_1} x_2^{i_2} \cdots x_k^{i_k} y^{i_{k+1}} b_{j_1,j_2,\dots,j_k,j_{k+1}} x_1^{j_1} x_2^{j_2} \cdots x_k^{j_k} y^{j_{k+1}} \, dX$$

$$\geqslant 0$$

$$= \int_{\|X\|=1} a_{i_1,i_2,\dots,i_k} x_1^{i_1} x_2^{i_2} \cdots x_k^{i_k} a_{j_1,j_2,\dots,j_k} x_1^{j_1} x_2^{j_2} \cdots x_k^{j_k} \, dX$$

$$+ y^{j_{k+1}} \int_{\|X\|=1} -a_{i_1,i_2,\dots,i_k} x_1^{i_1} x_2^{i_2} \cdots x_k^{i_k} b_{j_1,j_2,\dots,j_k,j_{k+1}} x_1^{j_1} x_2^{j_2} \cdots x_k^{j_k} \, dX$$

$$+ y^{i_{k+1}} \int_{\|X\|=1} -a_{j_1,j_2,\dots,j_k} x_1^{j_1} x_2^{j_2} \cdots x_k^{j_k} b_{i_1,i_2,\dots,i_k,i_{k+1}} x_1^{i_1} x_2^{i_2} \cdots x_k^{i_k} \, dX$$

$$+ y^{i_{k+1}} y^{j_{k+1}} \int_{\|X\|=1} b_{i_1,i_2,\dots,i_k,i_{k+1}} x_1^{i_1} x_2^{i_2} \cdots x_k^{i_k} b_{j_1,j_2,\dots,j_k,j_{k+1}} x_1^{j_1} x_2^{j_2} \cdots x_k^{j_k} \, dX$$

$$\geqslant 0$$

$$= c + 2c_i y^i + c_{ij} y^i y^j \geqslant 0;$$

where the c's are constants. This can be represented in matrix notation

$$Y'AY + B'Y + C \geqslant 0,$$

where A is a n x n symmetric matrix,

$$A_{ij} = \begin{cases} c_{ij}, & \text{for } i = j \\ \frac{1}{2} c_{ij}, & \text{for } i \neq j \end{cases}$$

B is a n x 1 matrix,

$$B_i = 2c_i$$

and C is a 1 x 1 matrix,

$$C = c.$$

Let

$$Y = TZ,$$

where T is a n x n matrix. Then

$$Y' = Z'T',$$

and

$$Y'AY + B'Y + C \geqslant 0$$

becomes

$$Z'T'ATZ + B'TZ + C \geqslant 0.$$

Choose T such that T'AT is a diagonal. This is possible since A is symmetric. $T'AT = D[\lambda_1, \lambda_2, \ldots, \lambda_n]$ where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A. None of the λ_i can be equal to zero because if the coefficient of a squared term were equal to zero and the coefficient of the linear term were not, Y could be increased in magnitude to the point where our inequality would not hold.

Since T'AT = D,

$$Z'T'ATZ + B'TZ + C \geq 0$$

becomes

$$Z'DZ + B'TZ + C \geq 0$$

$$\lambda_i (Z_i)^2 + (B'T)_i Z_i + C \geq 0$$

$$\left(\sqrt{\lambda_i}\, Z_i + \frac{(B'T)_i}{2\sqrt{\lambda_i}}\right)^2 + C - \frac{(B'T)_i^2}{4\lambda_i} \geq 0.$$

A unique minimum will exist only when

$$\left(\sqrt{\lambda_i}\, Z_i + \frac{(B'T)_i}{2\sqrt{\lambda_i}}\right)^2 = 0,$$

for all i. Therefore a unique minimum will exist when

$$Z_i = \frac{-(B'T)_i}{2\lambda_i},$$

which implies a unique minimum will exist at Y = TZ. Q. E. D.

Now that the ratio is defined, in the next section we will investigate by means of a particular example whether our defined ratio converges to the singularity of the function.

IV. Example

In this section the objective is to show that the desired generalization of König's theorem holds for a particular function of several real variables if the definition developed in the last section for the ratio of coefficients in the Taylor series expansion is used.

The example to be investigated is a function of two variables

$$h(x^1, x^2) = \frac{1}{(x^1-1)^2 + (x^2-1)^2},$$

which has its only singularity at the point (1,1). The Taylor series expansion for h is

$$h(X) = \sum_{n=0}^{\infty} \frac{1}{n!} h^{(n)}(0) X^n$$

where $h^{(n)}(0)$ is the n^{th} Fréchet derivative of the function h evaluated at (0,0) and X^n is understood to be $(\overbrace{X, X, \ldots, X}^{n\text{-times}})$. We are interested in constructing a sequence of Y_n where

$$Y_n = \frac{1}{n!} h^{(n)}(0) \Big/ \frac{1}{(n+1)!} h^{(n+1)}(0),$$

but in the last section we define this ratio Y_n as the value that minimizes

$$\int_{\|X\|=1} \left[\frac{1}{n!} h^{(n)}(0) X^n - \frac{1}{(n+1)!} h^{(n+1)}(0) X^n Y_n\right]^2 dX.$$

Referring to the lemma we proved in Section III, we find that we need to minimize

$$a + b y_n^1 + c y_n^2 + d(y_n^1)^2 + e(y_n^2)^2 + f y_n^1 y_n^2 \geq 0,$$

where, using summation notation,

$$a = \frac{1}{n!\,n!} \int_{\|X\|=1} h_{i_1,i_2,\ldots,i_n} x^{i_1} x^{i_2} \cdots x^{i_n} h_{j_1,j_2,\ldots,j_n} x^{j_1} x^{j_2} \cdots x^{j_n} \, dX,$$

$$b = \frac{-2}{n!(n+1)!} \int_{\|X\|=1} h_{i_1,i_2,\ldots,i_n} x^{i_1} x^{i_2} \cdots x^{i_n} h_{j_1,j_2,\ldots,j_n,1} x^{j_1} x^{j_2} \cdots x^{j_n} \, dX,$$

$$c = \frac{-2}{n!(n+1)!} \int_{\|X\|=1} h_{i_1,i_2,\ldots,i_n} x^{i_1} x^{i_2} \cdots x^{i_n} h_{j_1,j_2,\ldots,j_n,2} x^{j_1} x^{j_2} \cdots x^{j_n} \, dX,$$

$$d = \frac{1}{(n+1)!\,(n+1)!} \int_{\|X\|=1} h_{i_1,i_2,\ldots,i_n,1} x^{i_1} x^{i_2} \cdots x^{i_n} h_{j_1,j_2,\ldots,j_n,1} x^{j_1} x^{j_2} \cdots x^{j_n} \, dX,$$

$$e = \frac{1}{(n+1)!\,(n+1)!} \int_{\|X\|=1} h_{i_1,i_2,\ldots,i_n,2} x^{i_1} x^{i_2} \cdots x^{i_n} h_{j_1,j_2,\ldots,j_n,2} x^{j_1} x^{j_2} \cdots x^{j_n} \, dX,$$

$$f = \frac{2}{(n+1)!(n+1)!} \int_{\|X\|=1} h_{i_1,i_2,\ldots,i_n,1} x^{i_1} x^{i_2} \cdots x^{i_n} h_{j_1,j_2,\ldots,j_n,2} x^{j_1} x^{j_2} \cdots x^{j_n} \, dX,$$

and

$$h_{i_1, i_2, \ldots, i_n} = \left. \frac{\partial^n h}{\partial x^{i_1} \partial x^{i_2} \cdots \partial x^{i_n}} \right|_{0,0}$$

$$h_{i_1, i_2, \ldots, i_n, 1} = \left. \frac{\partial^{n+1} h}{\partial x^{i_1} \partial x^{i_2} \cdots \partial x^{n} \partial x^{1}} \right|_{0,0}$$

Rotating the axes so that the cross-product term drops out, then completing the square, we find that the minimum occurs at

$$y_n^1 = \frac{-b\left[(d-e) + \sqrt{f^2 + (d-e)^2}\right] - cf}{2\left[f^2 + (d-e)^2 + (d+e)\sqrt{f^2 + (d-e)^2}\right]}$$

$$+ \frac{-b\left[-(d-e) + \sqrt{f^2 + (d-e)^2}\right] + cf}{2\left[-f^2 - (d-e)^2 + (d+e)\sqrt{f^2 + (d-e)^2}\right]}$$

$$y_n^2 = \frac{bf - c\left[(d-e) + \sqrt{f^2 + (d-e)^2}\right]}{2\left[-f^2 - (d-e)^2 + (d+e)\sqrt{f^2 + (d-e)^2}\right]}$$

$$+ \frac{-bf - c\left[-(d-e) + \sqrt{f^2 + (d-e)^2}\right]}{2\left[f^2 + (d-e)^2 + (d+e)\sqrt{f^2 + (d-e)^2}\right]}$$

Because of symmetry,

$$b = c \qquad \text{and} \qquad d = e.$$

Hence,

$$y_n^1 = y_n^2 = \frac{-b}{f + 2d} .$$

We then know that the nth term in our sequence of ratios is

$$y_n^1 = y_n^2 = \Big[(n+1) \int_{\|X\|=1} h_{i_1,i_2,\ldots,i_n} h_{j_1,j_2,\ldots,j_n,1} x^{i_1} x^{i_2} \cdots$$

$$\cdots x^{i_n} x^{j_1} x^{j_2} \cdots x^{j_n}\, dX \Big] \Big/ \Big[\int_{\|X\|=1} h_{i_1,i_2,\ldots,i_n,1} h_{j_1,j_2,\ldots}$$

$$\ldots_{,j_n,1} x^{i_1} x^{i_2} \cdots x^{i_n} x^{j_1} x^{j_2} \cdots x^{j_n}\, dX + \int_{\|X\|=1} h_{i_1,i_2,\ldots}$$

$$\ldots_{,i_n,1} h_{j_1,j_2,\ldots,j_n,2}\, x^{i_1} x^{i_2} \cdots x^{i_n} x^{j_1} x^{j_2} \cdots x^{j_n}\, dX \Big].$$

After calculating the first few terms in the sequence, it appears that one might have

$$y_n^1 = y_n^2 = \frac{(n+1)}{(n+2)} .$$

If we can show that this is indeed the case, then we will have illustrated for this particular example that the ratio of coefficients in the Taylor series expansion, as we have defined them, do in fact converge to the singularity.

To show that

$$y_n^1 = \frac{(n+1)}{(n+2)} ,$$

we need to prove that

$$y_n' = \Big[(n+1)h_{i_1,i_2,\ldots,i_n}h_{j_1,j_2,\ldots,j_n,1}\int_{\|X\|=1} x^{i_1}x^{i_2}\cdots x^{i_n}x^{j_1}x^{j_2}\cdots$$

$$\cdots x^{j_n}\,dX\Big]\Big/\Big[h_{i_1,i_2,\ldots,i_n,1}\,(h_{j_1,j_2,\ldots,j_n,1}+h_{j_1,j_2,\ldots,j_n,2})\cdot \qquad (1)$$

$$\int_{\|X\|=1} x^{i_1}x^{i_2}\cdots x^{i_n}x^{j_1}x^{j_2}\cdots x^{j_n}\,dX\Big]$$

$$= \frac{(n+1)}{(n+2)}\,.$$

To show this equality term by term we need to show

$$\overset{\circ}{h}_{p+1,q} + \overset{\circ}{h}_{p,q+1} = (p+q+2)\,h_{p,q} \qquad (2)$$

where

$$p + q = n$$

and

$$\overset{\circ}{h}_{p,q} = \left.\frac{\partial^n h}{\partial x^{1^p}\partial x^{2^q}}\right|_{x^1=x^2=0}\,.$$

In developing the theory further the n^{th} derivative of h with respect to x^1 p times and with respect to x^2 q times will be denoted as $h_{p,q}$ or

$$h_{p,q} = \frac{\partial^n h}{\partial x^{1^p}\partial x^{1^q}}\,.$$

In trying to show that the relationship (2) is true, one is reminded of Euler's Relation for homogeneous functions which states that for a function $f(x^1, x^2)$,

$$x^1 \frac{\partial f}{\partial x^1} + x^2 \frac{\partial f}{\partial x^2} = k\left(f(x^1, x^2)\right),$$

where k is the degree of homogeneity. [4] Because of the form of the function h, another function g is defined such that

$$g(x^1, x^2) = h\left((x^1+1), (x^2+1)\right) = \frac{1}{(x^1)^2 + (x^2)^2}.$$

A function $f(x^1, x^2)$ is said to be homogeneous of degree k if

$$f(\lambda x^1, \lambda x^2) = \lambda^k f(x^1, x^2).$$

Therefore, $g(x^1, x^2)$ is homogeneous of degree -2 since

$$g(\lambda x^1, \lambda x^2) = \frac{1}{(\lambda x^1)^2 + (\lambda x^2)^2} = \frac{1}{\lambda^2}\left(\frac{1}{(x^1)^2 + (x^2)^2}\right)$$

$$= \lambda^{-2} g(x^1, x^2).$$

Applying Euler's Relation to $g(x^1, x^2)$

$$x^1 \frac{\partial g}{\partial x^1} + x^2 \frac{\partial g}{\partial x^2} = -2g.$$

But

$$\frac{\partial g(x^1, x^2)}{\partial x^1} = \frac{\partial h((x^1+1), (x^2+1))}{\partial x^1}$$

therefore

$$x^1 \frac{\partial h((x^1+1), (x^2+1))}{\partial x^1} + x^2 \frac{\partial h((x^1+1), (x^2+1))}{\partial x^2}$$

$$= -2h\left((x^1+1), (x^2+1)\right).$$

Replacing x^1 by $(x^1 - 1)$ and x^2 by $(x^2 - 1)$, then

$$(x^1-1)\frac{\partial h(x^1, x^2)}{\partial x^1} + (x^2-1)\frac{\partial h(x^1, x^2)}{\partial x^2}$$

$$= -2h(x^1, x^2). \tag{3}$$

Differentiating (3) with respect to x^1 yields

$$\frac{\partial h}{\partial x^1} + (x^1-1)\frac{\partial^2 h}{\partial x^{1^2}} + (x^2-1)\frac{\partial^2 h}{\partial x^1 \partial x^2} = -2\frac{\partial h}{\partial x^1}$$

or

$$(x^1-1)\frac{\partial^2 h}{\partial x^{1^2}} + (x^2-1)\frac{\partial^2 h}{\partial x^1 \partial x^2} = -3\frac{\partial h}{\partial x^1};$$

and because of symmetry

$$(x^1-1)\frac{\partial^2 h}{\partial x^1 \partial x^2} + (x^2-1)\frac{\partial^2 h}{\partial x^{2^2}} = -3\frac{\partial h}{\partial x^2}.$$

Evaluating these last two equations at (0,0) yields

$$\left.\frac{\partial^2 h}{\partial x^{1^2}}\right|_{0,0} + \left.\frac{\partial^2 h}{\partial x^1 \partial x^2}\right|_{0,0} = 3\left.\frac{\partial h}{\partial x^1}\right|_{0,0}$$

and

$$\left.\frac{\partial^2 h}{\partial x^1 \partial x^2}\right|_{0,0} + \left.\frac{\partial^2 h}{\partial x^{2^2}}\right|_{0,0} = 3\left.\frac{\partial h}{\partial x^2}\right|_{0,0}$$

which satisfy (2) for p = 1, q = 0 and p = 0, q = 1.

Continuing in this manner, it would appear that a general expression could be obtained which would probably be of the form

$$(x^1-1)\,h_{p+1,q} + (x^2-1)\,h_{p,q+1} = -(p+q+2)\,h_{p,q}. \qquad (4)$$

To prove that (4) holds, it is assumed true for a particular p and q and then shown that this implies it is true for (p + 1) and q. Because of symmetry in p and q this proves the relation for all p and q. Assuming that (4) is true, differentiation with respect to x^1 yields

$$h_{p+1,q} + (x^1-1)\,h_{(p+1)+1,q} + (x^2-1)\,h_{p+1,q+1}$$
$$= -(p+q+2)\,h_{p+1,q}$$

or

$$(x^1-1)\,h_{(p+1)+1,q} + (x^2-1)\,h_{p+1,q+1}$$
$$= -((p+1)+q+2)\,h_{p+1,q} \;;$$

but this last equality implies (4) is true for $(p + 1)$ and q, hence we have shown by induction that (4) is true.

If (4) is evaluated at $x^1 = x^2 = 0$, then

$$\overset{\circ}{h}_{p+1,q} + \overset{\circ}{h}_{p,q+1} = (p+q+2)\,\overset{\circ}{h}_{p,q}$$

but this is precisely what (2) states. Hence

$$y_n^1 = y_n^2 = \frac{(n+1)}{(n+2)} \;,$$

and the limit as $n \to \infty$ gives $y^1 = 1$, $y^2 = 1$ which agrees with the location of the singularity.

Not only have we shown that the generalized König's theorem holds for this particular example when the definition presented in Section III is used for the ratio, but we have shown that the convergence is independent of the norm over which the minimization is taken. In this process we have also shown that the theorem holds for functions of two variables of the form

$$h(x^1, x^2) = \frac{1}{(ax^1+b)^2 + (cx^2+d)^2}$$

which satisfy the conditions of the theorem, and where a, b, c and d are real, since by a transformation of the variables this can be expressed in the same form as our example.

V. Conclusions

In Section IV it was shown that the generalized König's theorem holds not only for the example but for functions of two variables of the form

$$h(x^1, x^2) = \frac{1}{(ax^1 + b)^2 + (cx^2 + d)^2}$$

which satisfy the conditions of the theorem and where a, b, c and d are real. It will be shown that this result can be extended to functions of m real variables. Consider a function of m real variables,

$$h(x^1, x^2, \cdots, x^m) = \frac{1}{f(x^1, x^2, \cdots, x^m)}$$

whose Taylor series expansion converges out to the nearest singularity. Here f is a positive-definite or negative-definite homogeneous function with degree of homogeneity 2, hence h is homogeneous of degree -2. f has the form

$$f(x^1, x^2, \cdots, x^m) = \sum_{i=1}^{m} b_{ii} x^i x^i + \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} b_{ij} x^i x^j .$$

The coefficients can be considered as elements of a symmetric matrix A where

$$A_{ij} = \begin{cases} b_{ii}, & \text{for } i = j \\ \frac{1}{2} b_{ij}, & \text{for } i \neq j \end{cases} .$$

Because of symmetry A can be transformed by a simularity transformation into a diagonal matrix. Replacing the x^i by their transformed values, we have

$$f(x^1, x^2, \cdots, x^m) = \sum_{i=1}^{m} \alpha_i x^i x^i \; .$$

Using another transformation, replacing x^i by $x^i/\sqrt{\alpha}$, we have

$$f(x^1, x^2, \cdots, x^m) = \sum_{i=1}^{m} x^i x^i \; .$$

This last transformation is possible because f is positive-definite. If f had originally been negative-definite, it could have been made positive-definite by multiplying by -1. If we now replace x^i by $(x^i - 1)$, we have

$$h(x^1, x^2, \cdots, x^m) = \frac{1}{(x^1-1)^2 + (x^2-1)^2 + \cdots + (x^m-1)^2} \; .$$

We next need to show that a function of this form will converge to $(\overbrace{1, 1, \cdots, 1}^{m\text{-times}})$ if we use our definition of the ratio of two successive coefficients in the Taylor series expansion.

In the general case the ratio Y_n is the value that minimizes

$$a + b y^{j_{n+1}} + c y^{i_{n+1}} + d y^{i_{n+1}} y^{j_{n+1}} \geq 0$$

where

$$a = \frac{1}{n!\,n!} h_{i_1, i_2, \ldots, i_n} h_{j_1, j_2, \ldots, j_n} \int_{\|X\|=1} x^{i_1} x^{i_2} \cdots$$

$$\cdots x^{i_n} x^{j_1} x^{j_2} \cdots x^{j_n} \, dX \; ,$$

$$b = \frac{-1}{m!(m+1)!} h_{i_1,i_2,\ldots,i_m} h_{j_1,j_2,\ldots,j_{m+1}} \int_{\|X\|=1} x^{i_1}x^{i_2}\cdots x^{i_m}x^{j_1}x^{j_2}\cdots x^{j_m}\,dX,$$

$$c = \frac{-1}{m!(m+1)!} h_{i_1,i_2,\ldots,i_{m+1}} h_{j_1,j_2,\ldots,j_m} \int_{\|X\|=1} x^{i_1}x^{i_2}\cdots x^{i_m}x^{j_1}x^{j_2}\cdots x^{j_m}\,dX,$$

$$d = \frac{1}{(m+1)!(m+1)!} h_{i_1,i_2,\ldots,i_m,i_{m+1}} h_{j_1,j_2,\ldots,j_m,j_{m+1}} \int_{\|X\|=1} x^{i_1}x^{i_2}\cdots x^{i_m}x^{j_1}x^{j_2}\cdots x^{j_m}\,dX.$$

Making use of the property of symmetry and that $y_n^1 = y_n^2 = \ldots = y_n^m = y_n$, we find that y_n is the value that minimizes

$$a + b y_n + c y_n^2 + d y_n^2 \geq 0$$

where

$$a = \frac{1}{m!\,m!} h_{i_1,i_2,\ldots,i_m} h_{j_1,j_2,\ldots,j_m} \int_{\|X\|=1} x^{i_1}x^{i_2}\cdots x^{i_m}x^{j_1}x^{j_2}\cdots x^{j_m}\,dX,$$

$$b = \frac{-2m}{m!(m+1)!} h_{i_1,i_2,\ldots,i_m} h_{j_1,j_2,\ldots,j_m,1} \int_{\|X\|=1} x^{i_1}x^{i_2}\cdots x^{i_m}x^{j_1}x^{j_2}\cdots x^{j_m}\,dX,$$

$$c = \frac{m}{(m+1)!^2} h_{i_1,i_2,\ldots,i_m,1} h_{j_1,j_2,\ldots,j_m,1} \int_{\|X\|=1} x^{i_1}x^{i_2}\cdots x^{i_m}x^{j_1}x^{j_2}\cdots x^{j_m}\,dX,$$

$$d = \frac{m^2-m}{(m+1)!^2} h_{i_1,i_2,\ldots,i_m,1} h_{j_1,j_2,\ldots,j_m,2} \int_{\|X\|=1} x^{i_1}x^{i_2}\cdots x^{i_m}x^{j_1}x^{j_2}\cdots x^{j_m}\,dX.$$

By completing the square, we find that the value of y_n that makes the integral a minimum is

$$y_n = \frac{-b}{2(c+d)}$$

$$= \left[(n+1) h_{i_1,i_2,\ldots,i_n} h_{j_1,j_2,\ldots,j_n,1} \int_{\|X\|=1} x^{i_1} x^{i_2} \ldots x^{i_n} x^{j_1} x^{j_2} \ldots x^{j_n} \, dX\right]$$

$$\Big/ \left[h_{i_1,i_2,\ldots,i_n,1} h_{j_1,j_2,\ldots,j_n,1} \int_{\|X\|=1} x^{i_1} x^{i_2} \ldots x^{i_n} x^{j_1} x^{j_2} \ldots x^{j_n} \, dX \right.$$

$$\left. + (n-1) h_{i_1,i_2,\ldots,i_n,1} h_{j_1,j_2,\ldots,j_n,2} \int_{\|X\|=1} x^{i_1} x^{i_2} \ldots x^{i_n} x^{j_1} x^{j_2} \ldots x^{j_n} \, dX \right].$$

If we can show that

$$y_n = \frac{(n+1)}{(n+2)}$$

then we would have our ratio converging to the nearest singularity. To show

$$y_n = \left[(n+1) h_{i_1,i_2,\ldots,i_n} h_{j_1,j_2,\ldots,j_n,1} \int_{\|X\|=1} x^{i_1} x^{i_2} \ldots x^{i_n} x^{j_1} x^{j_2} \ldots x^{j_n} \, dX\right]$$

$$\Big/ \left[h_{i_1,i_2,\ldots,i_n,1} \left(h_{j_1,j_2,\ldots,j_n,1} + (n-1) h_{j_1,j_2,\ldots,j_n,2} \right) \cdot \right.$$

$$\left. \int_{\|X\|=1} x^{i_1} x^{i_2} \ldots x^{i_n} x^{j_1} x^{j_2} \ldots x^{j_n} \, dX \right]$$

$$= \frac{(n+1)}{(n+2)};$$

term by term, we need only show

$$h^0_{p_1+1,p_2,\ldots,p_n} + (n-1) h^0_{p_1,p_2+1,p_2,\ldots,p_n} = (n+2) h^0_{p_1,p_2,\ldots,p_n}$$

where

$$p_1 + p_2 + \cdots + p_m = n$$

and

$$h^0_{p_1,p_2,\ldots,p_m} = \left.\frac{\partial^n h}{\partial x^{1^{p_1}} \partial x^{2^{p_2}} \cdots \partial x^{m^{p_m}}}\right|_0 ;$$

also,

$$h_{p_1,p_2,\ldots,p_m} = \frac{\partial^n h}{\partial x^{1^{p_1}} \partial x^{2^{p_2}} \cdots \partial x^{m^{p_m}}} .$$

Again Euler's Relation can be applied to yield

$$(x^1-1)\, h_{p_1+1,p_2,p_3,\ldots,p_m} + (x^2-1)\, h_{p_1,p_2+1,p_3,\ldots,p_m}$$

$$+ \cdots + (x^m-1) h_{p_1,p_2,\ldots,p_m} = -(n+2)\, h_{p_1,p_2,\ldots,p_m} ;$$

and because of symmetry

$$(x^1-1) h_{p_1+1,p_2,\ldots,p_m} + (m-1)(x^2-1)\, h_{p_1,p_2+1,p_3,\ldots,p_m}$$

$$= -(n+2)\, h_{p_1,p_2,\ldots,p_m}$$

when $x^2 = x^3 = \ldots = x^m$. Evaluating this at the origin gives

$$h^0_{p_1+1,p_2,\ldots,p_m} + (m-1)\, h^0_{p_1,p_2+1,p_3,\ldots,p_m} = (n+2)\, h^0_{p_1,p_2,\ldots,p_m}$$

which implies that

$$y_n = \frac{(n+1)}{(n+2)} ,$$

or that the ratio of coefficients converge to the value of the singularity $(\overbrace{1,1,\ldots,1}^{n\text{-times}})$. Again this convergence is independent of the norm over which the minimization is taken.

While the goal of generalizing König's theorem to functions of several variables has not been achieved, we have shown that it is true for a large class of functions. This thesis, however, shows that the first step in this generalization, properly defining the ratio, can be done in terms of Fréchet derivatives.

To prove König's theorem in the general case, one would have to resolve such problems as how to eliminate a singularity from a function, convergence, etc.

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UNCLASSIFIED

Security Classification

DOCUMENT CONTROL DATA - R&D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) Naval Postgraduate School Monterey, California 93940	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP

3. REPORT TITLE

König's Theorem and a Possible Generalization

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

Master of Science Thesis, March 1968

5. AUTHOR(S) (Last name, first name, initial)

Johnson, Patricia C.

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
March 1968	44	15

8a. CONTRACT OR GRANT NO. b. PROJECT NO. c. d.	9a. ORIGINATOR'S REPORT NUMBER(S) 9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)

10. AVAILABILITY/LIMITATION NOTICES

[redacted]

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY

13. ABSTRACT

The possibility of generalizing König's Theorem to functions of more than one variable is investigated. The generalized Taylor series expansion of a function of several variables is introduced, and the ratio of coefficients in the expansion is defined using Fréchet derivatives. It is shown that for a particular example of a function of two variables the generalized König's Theorem holds. The theorem is then shown to hold for a large class of functions.

DD FORM 1 JAN 64 1473

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14 KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
König's Theorem Fréchet Differentiation						

DD FORM 1 NOV 65 1473 (BACK)

S/N 0101-807-6821

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